On a Theorem of Ankeny and Rivlin

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(Dedicated to Late Professor G.M. Qazi)

Abstract—If \( p(z) \) is a polynomial of degree \( n \) and does not vanish in \( |z| < 1 \), then it was shown by Dewan, Hans and Kaur (2010) that

\[
\{M(p, R)\}^s \leq \left( \frac{R^{ns} + 1}{2} \right) \{M(p, 1)\}^s
\]

In this paper, we have improved the above inequality by involving some or all the coefficients of \( p(z) \).

Inequality (1.4) was independently proved by Qazi [5, Lemma 1], who also under the same hypothesis proved that

\[
\max_{|z|=1} |p'(z)| \leq n\Lambda
\]

where

\[
\Lambda = \left\{ \frac{1 + \left( \frac{\mu}{n} \right) a_0 |k^{\mu+1}|}{1 + k^{\mu+1} + \left( \frac{\mu}{n} \right) a_0 \left( (k^{\mu+1} + k^{2\mu}) \max_{|z|=1} |p(z)| \right)} \right\}
\]

The following result which is due to Gardner, Govil and Weems [6] is of independent interest, because it provides generalizations and refinements of inequalities (1.2), (1.3), (1.4) and (1.5).

**Theorem I.1.** If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu \) is a polynomial of degree \( n \), having no zeros in the disk \( |z| < k \), \( k \geq 1 \), then for \( 1 \leq \mu \leq n \)

\[
\max_{|z|=1} |p'(z)| \leq n \left\{ \frac{1 + \left( \frac{\mu}{n} \right) a_0 |k^{\mu+1}|}{1 + k^{\mu+1} + \left( \frac{\mu}{n} \right) a_0 \left( (k^{\mu+1} + k^{2\mu}) \max_{|z|=1} |p(z)| \right)} \right\} \times (\max_{|z|=1} |p(z)| - m).
\]

Clearly \( m = \min_{|z|=k} |p(z)|. \)

It was shown by Ankeny and Rivlin [7] that if \( p(z) \neq 0 \) in \( |z| < 1 \), then inequality (1.1) can be replaced by a sharper inequality.

**Theorem I.2.** If \( p(z) \) is a polynomial of degree \( n \), which does not vanish in \( |z| < 1 \), then

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) M(p, 1), \quad R \geq 1.
\]

Recently Dewan et al. [8] proved the following generalization as well as an improvement of Theorem B.

**Theorem I.3.** If \( p(z) \) is a polynomial of degree \( n \), having no zeros in \( |z| < 1 \), then for every positive integer \( s \)

\[
\{M(p, R)\}^s \leq \left( \frac{R^n + 1}{2} \right) \{M(p, 1)\}^s, \quad R \geq 1.
\]

In this paper we improve Theorem C, by involving some or all the coefficients of \( p(z) \). More precisely, we prove
Theorem I.4. If
\[ p(z) = a_0 + \sum_{\nu=1}^{n} a_{\nu} z^\nu, \quad (1 \leq \mu \leq n) \]
is a polynomial of degree \( |z| < k, k \geq 1 \), having no zeros in \( s \) then every positive integer \( s \), we have
\[
\{M(p, R)\}^s \leq \left[ 1 + \frac{\mu}{n} \left| \frac{a_{\mu}}{a_0} \right| (k^{\mu+1} + k^{2\mu}) \right] \times \{M(p, 1)\}^s \tag{1.8}
\]

Theorem I.5. If
\[ p(z) = a_0 + \sum_{\nu=1}^{n} a_{\nu} z^\nu, \quad (1 \leq \mu \leq n) \]
is a polynomial of degree \( n \) having no zeros in \( |z| < k, k \geq 1 \), then every positive integer \( s \), we have
\[
\{M(p, R)\}^s \leq \left[ 1 + \frac{\mu}{n} \left| \frac{a_{\mu}}{a_0} \right| (k^{\mu+1} + k^{2\mu}) \right] \times \{M(p, 1)\}^s \tag{1.9}
\]

II. PROOFS OF THE THEOREMS

Proof of Theorem 1: Let
\[ M(p, 1) = \max_{|z|=1} |p(z)| \]
Since \( p(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < 1 \), therefore, by inequality (1.5), we have
\[
|p'(z)| \leq n \left| \frac{1 + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})}{k^{\mu+1} + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \right| M(p, 1),
\]
for \( |z| = 1 \).

Now \( p'(z) \) is a polynomial of degree \( n-1 \), therefore, it follows by (1.1) that for all \( r \geq 1 \), and \( 0 \leq \theta < 2\pi \),
\[
|p'(re^{i\theta})| \leq n \left| \frac{1 + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})}{k^{\mu+1} + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \right| r^{n-1} M(p, 1). \tag{2.1}
\]

Also for each \( \theta, 0 \leq \theta < 2\pi \) and \( R \geq 1 \), we have
\[
\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s \leq \int_1^R \frac{d}{dt} \{p(te^{i\theta})\}^s dt = s \{p(Re^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta} dt.
\]
This implies
\[
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s \int_1^R |\{p(te^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta}| dt.
\]
Which on combining with inequality (1.1) and (2.1), we get
\[
\begin{aligned}
\left| \{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s \right| &\leq n s \left( \frac{1 + \frac{\mu}{n} \left| \frac{a_{\mu}}{a_0} \right| (k^{\mu+1} + k^{2\mu})}{1 + k^{\mu+1} + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \right) \\
&\times \{M(p, 1)\}^s - m(p, k).
\end{aligned}
\]

This completes the proof of Theorem 1.

Proof of Theorem 2: The proof of Theorem 2 follows on the same lines as that of Theorem 1 by using inequality (1.6) instead of (1.5). But for the sake of completeness we give a brief outline of the proof. Since \( p(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < 1 \), therefore, by inequality (1.6), we have
\[
\max_{|z|=1} |p'(z)| \leq n \left| \frac{1 + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})}{k^{\mu+1} + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \right| r^{n-1} M(p, 1), \tag{2.2}
\]

Now \( p'(z) \) is a polynomial of degree \( n-1 \), therefore, it follows by (1.1) that for all \( r \geq 1 \), and \( 0 \leq \theta < 2\pi \),
\[
|p'(re^{i\theta})| \leq n \left| \frac{1 + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})}{k^{\mu+1} + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \right| r^{n-1} M(p, 1). \tag{2.3}
\]

Also for each \( \theta, 0 \leq \theta < 2\pi \) and \( R \geq 1 \), we have
\[
\begin{aligned}
\left| \{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s \right| &\leq n s \left( \frac{1 + \frac{\mu}{n} \left| \frac{a_{\mu}}{a_0} \right| (k^{\mu+1} + k^{2\mu})}{1 + k^{\mu+1} + \frac{n}{n} \left| \frac{a_n}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \right) \\
&\times \{M(p, 1)\}^s - m(p, k).
\end{aligned}
\]
Which implies
\[ |p(Re^{i\theta})|^s \leq \left( \{ M(p, 1) \} \right)^s + R^n s \left\{ \frac{1 + \left( \frac{\mu}{n} \right) \frac{|a_\mu|}{|a_0|-m} k^{\mu+1}}{1 + k^{\mu+1} + \left( \frac{\mu}{n} \right) \frac{|a_\mu|}{|a_0|-m} (k^{\mu+1} + k^{2\mu})} \right\} \times \left\{ \{ M(p, 1) \}^s - m(p, k) \right\} \]

From which the proof of Theorem 2 follows.

REFERENCES