Some Fixed Point Results in Fuzzy Inner Product Spaces

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Abstract—In this paper, some properties of fuzzy inner product spaces are given and some fixed point results are established.

Keywords—Fuzzy inner product space, decomposition theorem, fixed point theorems.

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I. INTRODUCTION

The idea of real probabilistic inner product space introduced by Sklar [1] and after that some authors established this definition in different approach (for references please see [5], [6], [8], [10], [11], [15]). Mukherjee & Bag [12] redefine fuzzy real inner product space introduced by Goudarzi & Vaezpour [7] in order to establish a decomposition theorem from a fuzzy real inner product into a family of crisp inner products. Some recent research on fuzzy inner product spaces is carried out in [16]-[25]. In [14], the present authors try to give a concept of fuzzy inner product in complex field and study some properties. In this paper some observations and some fixed point theorems namely Browder-Petryshyn, D. de Figueiredo etc. are studied in fuzzy setting.

We have organized the paper in the following way: Preliminary results are given in Section II. In Section III, some basic results space are studied. In Section IV, various type of fixed point results in fuzzy inner product space are established.

II. SOME PRELIMINARIES

Definition 2.1 [2] Let X be a linear space over a field F( field of real / complex numbers ). A fuzzy subset N of X × X (R is the set of real numbers) is called a fuzzy norm on X if ∀x, u ∈ X and c ∈ F, following conditions are satisfied:

(N1) ∀t ∈ R with t ≤ 0, N(x, t) = 0;
(N2) (∀t ∈ R, t ≥ 0, N(x, t) = 1) if x = 0;
(N3) ∀t ∈ R, t > 0, N(cx, t) = N(x, 1/|c|) if c ≠ 0;
(N4) ∀s, t ∈ R, x, u ∈ X:
N(x + u, s + t) ≥ min{N(x, s), N(u, t)}
(N5) N(x, .) is a non-decreasing function of R and lim N(x, t) = 1.

The pair (X, N) will be referred to as a fuzzy normed linear space.

Definition 2.2 [2] Let (X, N) be a fuzzy normed linear space. A subset A of X is said to be bounded iff ∃ t > 0 and 0 < r < 1 such that N(x, t) > 1 − r ∀ x ∈ A.

Definition 2.3 [14] Let V be a linear space over F(R or C). Define µ : V × V × F → [0, 1] such that ∀x, y, z ∈ V, t ∈ F, the following conditions hold:

(CFI-1) µ(x, x, t) = 0 ∀ t ∈ F − (R + ∪ {0}).
(CFI-2) µ(x, x, t) = 1 ∀ t ∈ R+ iff x = θ.
(CFI-3) µ(x, y, t) = µ(y, x, t).
(CFI-4) For any scalar k,

\[
\mu(kx, y, t) = \begin{cases} 
1 - \mu(x, y, \frac{1}{k}) & \text{if } k \in \mathbb{R}^- \\
H(t) & \text{if } k = 0 \\
F(x, y, \frac{1}{t}) & \text{otherwise}
\end{cases}
\]

Where

\[
H(t) = \begin{cases} 
1 & \text{if } t \in \mathbb{R}^+ \\
0 & \text{otherwise}
\end{cases}
\]

Then µ is said to be a fuzzy inner product and (V, µ) is a fuzzy inner product space.

Definition 2.4 [4] Let (X, N) be a fuzzy normed linear space.

A mapping T : (X, N) → (X, N) is said to be fuzzy non-expansive if

N(T(x) − T(y), t) ≥ N(x − y, t) ∀ x, y ∈ X ∀ t ∈ R.

Proposition 2.1 [4] Let (X, N) be a fuzzy normed linear space satisfying (N6) and T : X → X be a fuzzy non-expansive mapping. Then T is a non-expansive mapping w.r.t. each α-norm of N, where α ∈ (0, 1).

Remark 2.1 [4] If N(x, .) is upper semicontinuous, then the converse of the Proposition 2.1 is also true.

Definition 2.5 [3] Let (X, N) be a fuzzy normed linear space and α ∈ (0, 1). A sequence \{x_n\} in X is said to be α-convergent in X if ∃ x ∈ X such that

lim_{n→∞}N(x_n - x, t) > α ∀ t > 0 and x is called the limit of \{x_n\}.

Proposition 2.2 [3] Let (X, N) be a fuzzy normed linear space satisfying (N6). If \{x_n\} be an α-convergent sequence in (X, N), then \|x_n - x\|_α → 0 as n → ∞ (|| ||_α denotes the α-norm of N).

Definition 2.6 [4] Let (X, N) be a fuzzy normed linear space. A subset F of X is said to be l-fuzzy closed if for
each $\alpha \in (0,1)$ and for any sequence $\{x_n\}$ in $F$ and $x \in X$, $(\lim_{n \to \infty} N(x_n - x, t) \geq \alpha \forall t > 0) \Rightarrow x \in F$.

**Theorem 2.1** [9] (Browder-Petryshyn). Let $H$ be a Hilbert space and $C$ be a closed, convex and bounded subset of $H$. If $f : C \to C$ is a nonexpansive mapping on $C$, then $f$ has a fixed point in $C$.

**Observation 3.1** Let $(V, \mu)$ be a fuzzy inner product space satisfying (CFI-7), (CFI-8), and (CFI-9) and $N$ be its induced fuzzy norm. The $\alpha$-norms derived from induced fuzzy norm and from $\alpha$-inner products, $\alpha \in (0, 1)$, are same.

**Proof.** Since $(V, \mu)$ satisfy (CFI-7), the induced fuzzy norm $N$ of $\mu$ given by:

$$N(cx, t) = \begin{cases} \mu(|c|x, |c|x, t^2) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now (CFI-8) gives $\mu(x, x, t^2) > 0 \forall t > 0 \Rightarrow x = 0 \Rightarrow N(x, t) = \mu(x, x, t^2) > 0 \forall t > 0 \Rightarrow x = 0$.

i.e $N$ satisfy (N6).

Therefore for $\alpha \in (0, 1)$

$$||x||_\alpha = \land \{t > 0 : N(x, t) \geq \alpha\}$$

So $||x||_\alpha^2 = \land \{(t^2 > 0 : \mu(x, x, t^2) \geq \alpha) \land \{s > 0 : \mu(x, x, s) \geq \alpha\}$

(3.1.1).

On the other hand since $\mu$ satisfies (CFI-9), we have

$$\langle x, x \rangle_\alpha = \land \{t \in R : \mu(x, x, t) \geq \alpha\}$$

Now $\mu(x, x, t) = 0 \forall t \leq 0 \Rightarrow \langle x, x \rangle = \land \{t > 0 : \mu(x, x, t) \geq \alpha\}$

(3.1.2).

From (3.1.1) and (3.1.2) we get

$$||x||_\alpha^2 = ||x||_{\alpha}^2$$

$$\Rightarrow ||x||_\alpha = ||x||_{\alpha} \forall \alpha \in (0, 1).$$
Observation 3.2. (CFI-7) is necessary but not sufficient.

Proof. Here we justify it by an example of a fuzzy inner product to show that it induces a fuzzy norm \( N \) without satisfying (CFI-7).

Example 3.1. Consider the inner product space \( (R^2, \langle \cdot, \cdot \rangle) \) where \( \langle x, y \rangle = a_1 a_2 + b_1 b_2 \) where \( x = (a_1, b_1), y = (a_2, b_2) \), \( a_1, b_1, a_2, b_2 \in R \).

Consider the fuzzy inner product as

\[
\mu(cx, y, t) = N(x, x, s) = \begin{cases} 
1 & \text{if } Rl t > Rl\langle x, y \rangle \text{ or } Im t \neq Im\langle x, y \rangle \\
\frac{1}{2} & \text{if } Rl t = Rl\langle x, y \rangle \text{ and } Im t = Im\langle x, y \rangle \\
0 & \text{if } Rl t < Rl\langle x, y \rangle \text{ and } Im t = Im\langle x, y \rangle.
\end{cases}
\]

For \( c = 0 \), \( \mu(cx, y, t) = H(t) \).

We have shown that \( (R^2, \mu) \) is a fuzzy inner product space (Example 3.3.[14]).

Now we show that \( \mu \) does not satisfy (CFI-7).

Proof. Take \( s = -1 \) and \( t = 1 \).

Then for \( x = (1, 0), y = (0, 1) \) we have

\[
Rl st = -1 < 0 = Rl\langle x, y \rangle.
\]

Therefore \( \mu(x, y, st) = 0 \).

Again \( \langle x, x \rangle = 1 \) and \( \langle y, y \rangle = 1 \).

So \( s^2 = 1 = \langle x, x \rangle \) and \( t^2 = 1 = \langle y, y \rangle \)

\[
\Rightarrow \mu(x, x, s^2) = \frac{1}{2} \text{ and } \mu(y, y, t^2) = \frac{1}{2}.
\]

\[
\Rightarrow \min\{\mu(x, x, s^2), \mu(y, y, t^2)\} = \frac{1}{2}.
\]

Therefore \( \mu(x, y, st) = \min\{\mu(x, x, s^2), \mu(y, y, t^2)\} = \frac{1}{2} \).

Thus \( \mu \) does not satisfy (CFI-7).

Now define \( N : R^2 \times R \to [0, 1] \) by

\[
N(cx, t) = \begin{cases} 
\mu(|c|x, |c|x, t^2) & \text{if } t > 0 \\
0 & \text{otherwise}
\end{cases}
\]

We shall show that \( N \) is a fuzzy norm in \( R^2 \).

We have for \( t > 0 \)

\[
N(cx, t) = \begin{cases} 
1 & \text{if } Rl t^2 > Rl|c|^2|\langle x, x \rangle|^2 \\
\frac{1}{2} & \text{if } Rl t^2 = Rl|c|^2|\langle x, x \rangle|^2 \\
0 & \text{if } Rl t^2 < Rl|c|^2|\langle x, x \rangle|^2
\end{cases}
\]

and for \( t \leq 0 \), \( N(cx, t) = 0 \).

i.e. for \( t > 0 \)

\[
N(cx, t) = \begin{cases} 
1 & \text{if } Rl t > Rl|c||\langle x, x \rangle| \\
\frac{1}{2} & \text{if } Rl t = Rl|c||\langle x, x \rangle| \\
0 & \text{if } Rl t < Rl|c||\langle x, x \rangle|
\end{cases}
\]

and for \( t \leq 0 \), \( N(cx, t) = 0 \).

(N1) By definition for \( t \leq 0 \), \( N(x, t) = 0 \).

(N2) \( N(x, t) = 1 \) \( \forall t > 0 \)

\[\Rightarrow t > ||x|| \\forall t > 0 \]

\[\Rightarrow ||x|| = 0 \]

\[\Rightarrow x = \theta.\]

(N3) For \( t \leq 0 \), \( N(cx, t) = 0 = N(x, \frac{1}{|c|}) \) and for \( t > 0 \),

\[N(cx, t) = 1 \Rightarrow Rl t > Rl|c||x|| \Rightarrow Rl \frac{t}{|c|} > Rl|x|| \Rightarrow N(x, \frac{t}{|c|}) = 1.\]

Similarly if \( N(cx, t) = \frac{1}{2} \) then \( N(x, \frac{t}{|c|}) = \frac{1}{2} \) and \( N(cx, t) = 0 \) then \( N(x, \frac{t}{|c|}) = 0 \).

Thus \( N(cx, t) = N(x, \frac{t}{|c|}).\)

(N4) For \( s + t \leq 0 \) there is nothing to prove.

Let \( s + t > 0 \).

\[N(x + y, s + t) = 1 \implies N(x + y, s + t) \geq N(x, s) \land N(y, t).\]

\[N(x + y, s + t) = \frac{1}{2} \Rightarrow s + t = ||x + y|| \leq ||x|| + ||y||.\]

Then \( N(x, s) \land N(y, t) = 0 \) or \( \frac{1}{2} \).

\[\Rightarrow N(x + y, s + t) \geq N(x, s) \land N(y, t).\]

Similarly for \( N(x + y, s + t) = 0 \) we get

\[N(x + y, s + t) \geq N(x, s) \land N(y, t).\]

(N5) This follows from \( N(x, t) = 1 \) for \( t > ||x||.\)

IV. SOME FIXED POINT THEOREMS

In this Section Browder-Petryshyn and D.de Figueiredo theorem have been studied in fuzzy setting. Some other fixed point theorems also establish in this Section.

Definition 4.1. Let \( (X, \mu) \) be a fuzzy inner product space satisfying (CFI-7) and \( N \) be its induced fuzzy norm.

Then (i) \( (H, N) \) is said to be fuzzy Hilbert space if \( (H, N) \) is complete fuzzy normed linear space.

(ii) \( (H, N) \) is said to be \( \alpha \)-complete fuzzy inner product space if \( (H, N) \) is \( \alpha \)-complete fuzzy normed linear space.

(iii) \( C \subset X \) is said to be \( l \)-fuzzy closed if \( C \) is \( l \)-fuzzy closed w.r.t. \( N \).

(iv) \( C \subset X \) is said to be bounded if \( C \) is bounded w.r.t. \( N \).

(v) \( C \subset X \) is said to be \( l \)-fuzzy bounded if \( C \) is \( l \)-fuzzy bounded w.r.t. \( N \).

(vi) \( (H, \mu) \) is said to be \( l \)-fuzzy complete inner product space if \( (H, N) \) is \( l \)-fuzzy complete fuzzy normed linear space.

Remark 4.1. If \( (X, \mu) \) is a fuzzy inner product space satisfying (CFI-7) and (CFI-8), then induced fuzzy norm \( N \) of \( \mu \) satisfy (N6).

Proof. Since \( (X, \mu) \) is a fuzzy inner product space satisfying (CFI-7), then we get induced norm \( N \) as

\[
N(cx, t) = \begin{cases} 
\mu(|c|x, |c|x, t^2) & \text{if } t > 0 \\
0 & \text{otherwise}
\end{cases}
\]

From above it follows that \( \forall t > 0, \mu(x, x, t) > 0 \Leftrightarrow N(x, t) > 0.\)

Now by (CFI-8),

\[
\mu(x, x, t) > 0 \quad \forall t > 0 \Rightarrow x = \theta
\]

i.e. \( N(x, t) > 0 \quad \forall t > 0 \Rightarrow x = \theta
\]

i.e. \( N \) satisfy (N6).

Theorem 4.1. (Browder-Petryshyn). Let \( (H, \mu) \) be an \( l \)-fuzzy complete inner product space satisfying (CFI-7), (CFI-8), (CFI-9) and \( C \) be an \( l \)-fuzzy closed, convex and bounded.
set in $H$. If $f : C \to C$ is a fuzzy nonexpansive mapping on $C$, then $f$ has a fixed point in $C$.

**Proof.** Since $(H, \mu)$ is $l$-fuzzy complete inner product space satisfying (CFI-7), (CFI-8), (CFI-9) and then for each $\alpha \in (0, 1)$, $(H, \langle \cdot, \cdot \rangle_\alpha)$ are Hilbert space. Since $C$ is $l$-fuzzy closed then it is closed in $(H, \langle \cdot, \cdot \rangle_\alpha)$, $\alpha \in (0, 1)$ and $f$ is fuzzy nonexpansive implies $f$ is nonexpansive in $(H, \langle \cdot, \cdot \rangle_\alpha)$, $\alpha \in (0, 1)$. Now $C$ is bounded
\[ \Rightarrow \exists r \text{ and } t \text{ such that } N(x, t) > 1 - r \ \forall x \in H \]
\[ \Rightarrow \| x \|_{1-r} \leq t \ \forall x \in H \]
\[ \Rightarrow \| x \|_{\alpha_0} \leq t \ (\text{take } 1 - r = \alpha_0). \]
Therefore $C$ is bounded w.r.t $\alpha_0$-fuzzy norm and hence $C$ is bounded in $(H, \langle \cdot, \cdot \rangle_\alpha)$ in which $f$ is nonexpansive.

Thus we have by Theorem 2.2, $f$ has a fixed point in $C$.

**Theorem 4.2.** Let $(H, \mu)$ be an $l$-fuzzy complete inner product space satisfying (CFI-7), (CFI-8), (CFI-9) and $C$ be an $l$-fuzzy closed, convex and bounded set in $H$. Let $\mu(x, x, \cdot)$ be upper semicontinuous for each $x \in H$. If $f : C \to C$ is fuzzy nonexpansive then $F(f)$, the set of fixed point of $f$ is nonempty, where $f(x) = sx + (1-s)f(x) ; s \in (0, 1)$. Moreover $f_s$ have the same fixed points as of $f$.

**Proof.** Since $x, f(x) \in C$ and $C$ is convex set for $s \in (0, 1)$, $f_s(x) = sx + (1-s)f(x) \in C$.

Thus $f_s : C \to C$. Now for each $\alpha \in (0, 1)$ we have
\[ \| f_s(x_2) - f_s(x_1) \|_\alpha = \| s(x_2) + (1-s)f(x_2) - s(x_1) + (1-s)f(x_1) \|_\alpha \]
\[ \leq s \| |x_2 - x_1| \|_\alpha + (1-s) \| f(x_2) - f(x_1) \|_\alpha \]
\[ \leq s \| |x_2 - x_1| \|_\alpha + (1-s) \| x_2 - x_1 \|_\alpha \] (since $f$ is nonexpansive).

Thus $f_s : s \in (0, 1)$ is nonexpansive w.r.t. each $\alpha$-norm. Since $N(x, \cdot) = \mu(x, x, \cdot), x \in H$ is upper semicontinuous, it follows that $f_s$ is fuzzy nonexpansive (by Remark 2.1).

Therefore by Theorem 2.2, we have $f_s$ have a fixed point for each $s \in (0, 1)$ and hence $F(f_s) : s \in (0, 1)$ is nonempty.

Let $x_0$ be a fixed point of $f_s$.

Therefore $f_s(x_0) = sx_0 + (1-s)f(x_0) = sx_0 + (1-s)x_0 = x_0$.

Thus $x_0$ is also a fixed point of $f_s, s \in (0, 1)$.

**Definition 4.2.** Let $(H, \mu)$ be an $l$-fuzzy complete inner product space satisfying (CFI-7). The space $H$ is said to have fixed point property(f.p.p) if for any sectional fuzzy continuous function $f : H \to H$, there exist $x_0 \in X$ such that $f(x_0) = x_0$.

**Theorem 4.3.** Let $K$ be an $l$-fuzzy closed and convex set in a real $l$-fuzzy complete inner product space satisfying (CFI-7), (CFI-8). Then $K$ is fuzzy bounded if $K$ has f.p.p. for fuzzy nonexpansive mapping.

**Proof.** Since $(H, \mu)$ is real $l$-fuzzy complete inner product space satisfying (CFI-7), (CFI-8) then for each $\alpha \in (0, 1)$, $(H, \langle \cdot, \cdot \rangle_\alpha)$ are real Hilbert spaces. Since $K$ is $l$-fuzzy closed it is closed w.r.t. each $\alpha$-norm. Now $K$ has f.p.p.

\[ \Rightarrow \exists \alpha_0 \in (0, 1) \text{ such that } f(x_0) = x_0. \]

\[ \Rightarrow \exists \alpha_0 \in (0, 1) \text{ such that } f : K \to K \text{ is continuous w.r.t. } \| \|_{\alpha_0} \text{ (by Theorem 2.6) and } f(x_0) = x_0. \]

Since $f$ is a fuzzy nonexpansive $f$ is nonexpansive w.r.t. every $\alpha$-norm, for each $\alpha \in (0, 1)$. Then by Theorem 2.5, $K$ is bounded w.r.t. $\alpha_0$-norm.

So $\| x \|_{\alpha_0} \leq M \ \forall x \in K$

\[ \Rightarrow N(x, M) \geq \alpha_0 \ \forall x \in K \]
\[ \Rightarrow N(x, M) > \frac{\alpha_0}{t}. \]

Take $r = 1 - \frac{\alpha_0}{t}$. Then $N(x, M) > 1 - r$.

Thus $K$ is fuzzy bounded.

**Theorem 4.4.** (D. de Figueiredo). Let $(H, \mu)$ be an $l$-fuzzy complete inner product space satisfying (CFI-7), (CFI-8), (CFI-9) and $C$ be an $l$-fuzzy closed, convex and fuzzy bounded set in $H$ containing $\theta$. If $T : C \to C$ is a fuzzy nonexpansive mapping, then for any $x_0 \in C$ the sequence $\{x_n\}$, with
\[ x_n = T^n x_{n-1}, \quad n = 1, 2, 3, ... \text{ and } T_n x = \frac{n}{n+1} T x, \]
\[ \exists \alpha_0 \in (0, 1) \text{ and } \exists x \in F(T) \text{ (set of fixed point of } T) \text{ such that for any } t > 0, \exists M(t) \text{ such that } \mu(x_n - x', x_n - x', t) \geq \alpha_0 \ \forall n \geq M(t). \]

**Proof.** Since $(H, \mu)$ is $l$-fuzzy complete inner product space satisfying (CFI-7), (CFI-8), (CFI-9) and then for each $\alpha \in (0, 1)$, $(H, \langle \cdot, \cdot \rangle_\alpha)$ are Hilbert spaces. Since $C$ is $l$-fuzzy closed so it is closed in $(H, \langle \cdot, \cdot \rangle_\alpha)$, $\alpha \in (0, 1)$ and $T$ is fuzzy nonexpansive implies $f$ is nonexpansive in $(H, \langle \cdot, \cdot \rangle_\alpha), \alpha \in (0, 1)$.

Now $C$ is fuzzy bounded.

\[ \Rightarrow \exists r \text{ and } t \text{ such that } N(x, t) > 1 - r \ \forall x \in H \]
\[ \Rightarrow \| x \|_{1-r} \leq t \ \forall x \in H \]
\[ \Rightarrow \| x \|_{\alpha_0} \leq t \ (\text{take } 1 - r = \alpha_0). \]

Therefore $C$ is bounded w.r.t $\alpha_0$-fuzzy norm and hence $C$ is bounded in $(H, \langle \cdot, \cdot \rangle_{\alpha_0})$.

Then by Theorem 2.3, we have $\{x_n\}$ converges to a fixed point of $T$ w.r.t $\alpha_0$-inner product.

Let $x' \in F(T)$ where $\{x_n\}$ converges w.r.t $\alpha_0$-inner product

\[ \| x_n - x' \|_{\alpha_0} \to 0 \text{ as } n \to \infty. \]
\[ \| x_n - x' \|_{\alpha_0} \to 0 \text{ as } n \to \infty. \]
\[ \langle x_n - x', x_n - x' \rangle_{\alpha_0} \to 0 \text{ as } n \to \infty. \]

Thus for each $t > 0, \exists M(t)$ such that
\[ \mu(x_n - x', x_n - x', t) \geq \alpha_0 \ \forall \alpha_0 \ \forall n \geq M(t). \]

**Definition 4.3.** Let $(X, N)$ be an $l$-fuzzy complete normed linear space and $C$ be an $l$-fuzzy closed convex subset of $X$. Then $f : C \to X$ is said to be fuzzy demicompact if it has the property that whenever $\{x_n\}$ is a fuzzy bounded sequence in $C$ and there is $\alpha \in (0, 1)$, for which for any $t > 0, \exists M(t)$ such that $N(f(x_n) - x_n, t) \geq \alpha \ \forall n \geq M(t)$, then $\exists$ a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is convergent.

**Proposition 4.1.** Let $(X, N)$ be an $l$-fuzzy complete normed linear space satisfying (N6). Let $C$ be an $l$-fuzzy closed, convex subset of $X$ and $f : C \to X$ be fuzzy demicompact. Then $f$ is demicompact w.r.t. $\| \|_{\alpha_0} \forall \alpha \in (0, 1)$ where $\| \|_{\alpha_0}$ is an $\alpha$-norm of $N$.

**Proof.** Since $(X, N)$ be an $l$-fuzzy complete normed linear space satisfying (N6), so $X$ is complete w.r.t. $\langle \cdot \rangle_{\alpha_0}$ for each $\alpha \in (0, 1)$. Again since $C$ is $l$-fuzzy closed thus $C$ is closed w.r.t. $\| \|_{\alpha_0}$ for each $\alpha \in (0, 1)$.

Let $\{x_n\}$ be a bounded sequence in $C$ and $\{f(x_n) - (x_n)\}$ is
strongly convergent w.r.t. $||\cdot||_{\alpha_0}$ for $\alpha_0 \in (0, 1)$. 
Thus $\exists M$ such that $||x_n||_{\alpha_0} < M \ \forall n$
\[\Rightarrow N(x_n, M) \geq \alpha_0 \ \forall n\]
\[\Rightarrow N(x_n, M) \geq \alpha_0 > 1 - \beta_0 \ \forall n\]
\[\Rightarrow \{x_n\} \text{ fuzzy bounded.} \]
Again suppose $x \in X$ such that $\lim_{t \to 0} ||f(x_n) - x_n - x||_{\alpha_0} = 0$
\[\Rightarrow \text{for each } t > 0, \exists K(t) \text{ such that } ||f(x_n) - x_n - x||_{\alpha_0} < t\]
\[\Rightarrow N(f(x_n) - x_n - x, t) \geq \alpha_0 \ \forall n \geq K(t)\].
Since $f$ is fuzzy demicompact, it follows that $\exists$ a subsequence
\[\{x_{n_k}\}\] of $\{x_n\}$ which is convergent.

**Theorem 4.5.** Let $f : C \to C$ be fuzzy nonexpansive and fuzzy demicompact on the $l$-fuzzy closed, convex and $l$-fuzzy bounded set $C$ in an $l$-fuzzy complete inner product space $(H, \mu)$ satisfying (CFI-7), (CFI-8), (CFI-9) or strong strictly convex fuzzy normed linear space $(H, N)$ satisfying(N6). Then $F(f)$ is a nonempty, and convex set. Also each $s \in (0, 1)$, the sequence \[\{x_n\}\] where $x_n = sf(x_{n-1}) + (1-s)x_{n-1}, n = 1, 2, 3, ..., \text{ converges strongly to a fixed point of } f \text{ w.r.t. } ||\cdot||_{\alpha_0} \forall \alpha_0 \in (0, 1)$ and hence \[\{x_n\}\] is convergent w.r.t. $N$-norm ( norm of strong strictly convex space or induced fuzzy norm from $\mu$).

V. CONCLUSIONS

In recent past, numerous papers have been published in fuzzy functional analysis. But only a few works have been done on fuzzy inner product spaces. In this paper some properties of fuzzy inner product spaces are studied and some fixed point theorems are established in such spaces. There is a wide scope of research in this field and we hope that researchers will be benefitted through our work. Applications of fuzzy inner product spaces in real-world problems may be studied as a future scope of research.

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